Multichannel generalization of eigen-phase preserving supersymmetric transformations

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Abstract. We generalize eigen-phase preserving (EPP) supersymmetric (SUSY) transformations to $N > 2$ channel Schrödinger equation with equal thresholds. It is established that EPP SUSY transformations exist only in the case of even number of channels, $N = 2M$. A single EPP SUSY transformation provides an $M(M-1)+2$ parametric deformation of the matrix Hamiltonian without affecting eigen-phase shifts of the scattering matrix.

1. Introduction

In this paper we study N channel radial Schrödinger equation with equal thresholds. Such equation may describe scattering of particles with internal structure, for instance, spin [1, 2, 3]. Supersymmetric (SUSY) transformations allow analytical studies of Schrödinger equation with a wide class of interaction potentials [4, 5, 6, 2]. In particular, the inverse scattering problem [8, 9] for two-channel Schrödinger equation with equal thresholds may be treated by combined usage of single channel SUSY transformations [10] and eigen-phase preserving (EPP) supersymmetric (SUSY) transformations [11]. These transformations conserve the eigenvalues of the scattering matrix and modify its eigenvectors (coupling between channels), in a contrast to phase-equivalent SUSY transformations which do not modify scattering matrix at all [12, 13, 14, 15].

In [3] the two-channel neutron-proton potential was reproduced by a chain of SUSY transformations, where the coupled channel inverse scattering problem were decomposed into the fitting of the channel phase shifts [10, 16] and the fitting of the mixing between channels. The fitting of the mixing between channels was provided by the EPP SUSY transformations.

This paper extends the two-channel EPP SUSY transformations to higher number of channels.

The paper is organized as follows. We first fix our notations and recall basics of SUSY transformations [6, 17, 18, 19]. Given the explicit form of a second-order SUSY transformation operator we study the physical sector of SUSY transformations between real and symmetric Hamiltonians. We analyze the most general form of a second-order SUSY transformation for the case of mutually conjugated factorization energies. Then we discuss applications of SUSY transformations to the scattering problems and calculate how S-matrix transforms.

We start section 3 re-examining the conservation of its eigenvalues in the two-channel case. There is the following asymptotic condition for EPP SUSY transformations. The term with first-order derivative in the operator of EPP SUSY transformation vanishes at large distances.

To generalize EPP SUSY for arbitrary number of channels we study the matrix equation which comes from this asymptotic condition. We find that EPP SUSY transformations may exist for $2M$-channels only and obtain their general form. A 4-channel example explicitly shows how EPP SUSY transformations acts. We conclude with a summary of the obtained results and discussions of possible applications.

2. Second order SUSY transformations

2.1. Definition of SUSY transformations

SUSY transformations of stationary matrix Schrödinger equation are well known [17, 19]. In this subsection we just fix our notations.

Consider a family of matrix Hamiltonians $\mathbb{H} = \{H_a\}$

$$H_a = -I_N \frac{d^2}{dr^2} + V_a(r) \,, \tag{1}$$

where I_N is the $N \times N$ identity matrix, $V_a(r)$ is the $N \times N$ real symmetric matrix potential. A multi-index a parameterizes a family of potentials $V_a(r)$. A matrix Hamiltonian defines the system of ordinary differential equations

$$H_a \varphi_a(k, r) = k^2 \varphi_a(k, r) \,. \tag{2}$$

on $N \times N$ matrix functions $\varphi_a(k, r)$.

A (polinomial) SUSY transformation of (2) is a map of solutions

$$L_{ba} : \varphi_a(k, r) \to \varphi_b(k, r) = L_{ba} \varphi_a(k, r) \,, \tag{3}$$

provided by a differential matrix operator

$$L_{ba} = A_n \frac{d^n}{dr^n} + A_{n-1} \frac{d^{n-1}}{dr^{n-1}} + \ldots + A_1 \frac{d}{dr} + A_0 \,, \tag{4}$$

where A_j, $j = 0, \ldots, n$, are some matrix valued functions. This differential matrix operator obeys the intertwinning relation

$$L_{ba} H_a = H_b L_{ba} \,. \tag{5}$$

The intertwinning relation (5) defines both the operator L_{ab} and the transformed Hamiltonian H_b. We will consider a family of Hamiltonians $\mathbb{H}[H_a] = \{H_b | L_{ba} H_a = H_b L_{ba}\}$ related with the given Hamiltonian H_a by transformation operator (4). In the next subsection we present explicit form of the second order transformation operator and the transformed Hamiltonian.

2.2. Second-order SUSY algebra of matrix Schrödinger equation

Given initial matrix Hamiltonian H_0, we choose two $N \times N$ matrix solutions

$$H_0 u_j = E_j u_j \,, \qquad j = 1, 2 \,, \tag{6}$$

with E_1, E_2 called factorization constants. These functions determine a second order operator $L_{20}[u_1, u_2]$:

$$L_{20} f(r) = \left[I_N \partial_r^2 - V_0 + E_1 + (E_2 - E_1)(w_1 - w_2)^{-1}(w_1 - \partial_r) \right] f(r) \,, \tag{7}$$

where

$$w_j(r) = u_j'(r)u_j^{-1}(r)\,, \qquad w_j^2 + w_j' + E_j = V_0\,, \qquad j = 1,2\,. \tag{8}$$

Functions $w_j(r)$ are called superpotentials. We also introduce a second-order superpotential

$$W_2(r) = (E_2 - E_1)\,[w_1(r) - w_2(r)]^{-1}\,. \tag{9}$$

The more symmetric and compact form of formula (7) reads

$$L_{20}f(r) = \left[-H_0 + \frac{E_2 + E_1}{2} + W_2\left(\frac{w_1 + w_2}{2} - \partial_r\right)\right] f(r)\,. \tag{10}$$

Operator L_{20} and Hamiltonian H_0 obey the following algebra

$$L_{20}H_0 = H_2 L_{20}\,, \qquad H_2 = H_0 - 2W_2'\,, \tag{11}$$

$$L_{20}^\dagger L_{20} = (H_0 - E_1)(H_0 - E_2)\,, \quad L_{20}L_{20}^\dagger = (H_2 - E_1)(H_2 - E_2)\,. \tag{12}$$

The new (transformed) potential is expressed in terms of second-order superpotential W_2 as follows

$$V_2 = V_0 - 2W_2'\,. \tag{13}$$

The transformation operator has a global symmetry

$$L_{20}[u_1, u_2] = L_{20}[u_1(r)U_1, u_2U_2]\,, \qquad \det U_{1,2} \neq 0\,. \tag{14}$$

The second order SUSY transformations of an initial Hamiltonian H_0 form the family $\mathbb{H}_2[H_0] = \{H_2 | L_{20}H_0 = H_2L_{20}\}$. We will work only with Hamiltonians from $\mathbb{H}_2[H_0]$ and we omit subscripts in the notation of transformation operator $L_{20} \to L$.

2.3. Restrictions to the SUSY transformations

We restrict our consideration only to second order SUSY transformations with mutually conjugated factorization energies $E_1 = E_2^* = \mathcal{E}$. Potentials V_0 and V_2 are supposed to be real and symmetric. Hence the transformation functions u_1 and u_2 have to be mutually conjugated, $u_1 = u_2^* = u$. The symmetry of V_2 demands the symmetry of superpotentials (8), $w_1^T = w_1 = w$, $w_2^T = w_2 = w^*$. Defining the Wronskian of two matrix functions as

$$\mathrm{W}[u_1, u_2](r) \equiv u_1^T(r)u_2'(r) - u_1^{T\prime}(r)u_2(r) \tag{15}$$

$$= u_1^T(r)\left[w_2(r) - w_1^T(r)\right]u_2(r). \tag{16}$$

we see that the symmetry of superpotential w implies a vanishing self-Wronskian $\mathrm{W}[u, u] = 0$ of transformation functions [2].

We present the second-order superpotential W_2 in terms of the matrix Wronskian for further needs,

$$W_2(r) = (E_1 - E_2)u_2(r)\mathrm{W}[u_1, u_2]^{-1}(r)u_1^T(r)\,. \tag{17}$$

To specify acceptable choice of transformation solutions explicitly, we choose the basis in the solution space. Natural basis for the radial problem, $r \in (0, \infty)$, is formed by the Jost solutions $f(\pm k, r)$ with the exponential asymptotic behavior

$$f(k, r \to \infty) \to I_N e^{ikr}\,. \tag{18}$$

Let us expand the transformation functions in the Jost basis

$$u(r) = f_0(-K, r)C_j + f_0(K, r)D\,, \tag{19}$$

where $K = k_r + ik_i$, $K^2 = \mathcal{E}$, $k_i > 0$. Complex constant matrices C and D should provide vanishing self-wronskian $\mathrm{W}[u, u] = 0$. The wronskian of two solutions with the same k is a constant. For instance, $\mathrm{W}[f(-k, r), f(k, r)] = 2ikI_N$. Then, calculating $\mathrm{W}[u, u]$ we get a constraint on the possible choice of matrices C and D,

$$D^T C = C^T D\,. \tag{20}$$

Matrices C and D have an ambiguity due to symmetry (14). Rank of matrix C, $\mathrm{rank}C = M \leq N$, determines the structure of transformation operator. The sum of ranks $\mathrm{rank}C + \mathrm{rank}D \geq N$, otherwise operator L is undefined. Using (14) we may transform C to the form, where only first M columns are non-zero and linearly independent. Reordering channels (by permutations of rows in the system of equations (2)) we can put nontrivial $M \times M$ minor of C into the upper left corner. Then, C and D obey the following canonical form,

$$C = \begin{pmatrix} I_M & 0 \\ Q & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} X & -Q^T \\ 0 & I_{N-M} \end{pmatrix}, \tag{21}$$

where $X = X^T$ is a symmetric $M \times M$ complex matrix, and Q is $(N-M) \times M$ complex matrix. This canonical form is a gauge which fixes ambiguity (14) of transformation solutions.

2.4. Application to the scattering theory

In concrete physical applications of SUSY transformations we may further restrict the class of Hamiltonians. In particular, in scattering theory [1] we work with the radial problem, $r \in (0, \infty)$. The interaction potentials decrease sufficiently fast at large distances and may contain centrifugal term

$$\lim_{r\to\infty} r^2 V(r) = l(l + I_N)\,, \qquad l = \mathrm{diag}(l_1, \ldots, l_N)\,, \qquad \mathrm{e}^{il\pi} = \pm I_N\,. \tag{22}$$

The physical solution has the following asymptotic behavior

$$\psi(k, r \to \infty) \propto k^{-1/2} \left[\mathrm{e}^{-\mathrm{i}kr} \mathrm{e}^{\mathrm{i}l\frac{\pi}{2}} - \mathrm{e}^{\mathrm{i}kr} \mathrm{e}^{-\mathrm{i}l\frac{\pi}{2}} S(k)\right], \tag{23}$$

where matrix coefficient $S(k)$ is the scattering matrix.

Scattering matrix is related with the Jost matrix

$$S(k) = e^{il\frac{\pi}{2}} F(-k) F^{-1}(k) e^{il\frac{\pi}{2}}\,, \tag{24}$$

where the Jost matrix reads

$$F(k) = \lim_{r\to 0} \left[f^T(k, r) r^{\nu}\right] \left[(2\nu - 1)!!\right]^{-1}. \tag{25}$$

Diagonal matrix ν indicates the strength of the singularity in the potential near the origin

$$V(r \to 0) = \nu(\nu + I_N) r^{-2} + \mathrm{O}(1)\,. \tag{26}$$

Knowledge of the Jost solutions allows one to define scattering matrix. Supersymmetric transformations of Hamiltonian and solutions induce the transformation of scattering matrix. Formal approach to the calculations of the S-matrices was developed in the work of Amado [20].

Let us consider how the Jost solution transforms asymptotically,

$$\begin{aligned}
&(Lf_0)(k, r \to \infty) = \\
&= \left[-k^2 + \frac{E_2 + E_1}{2} + \left(W_2 \frac{w_1 + w_2}{2}\right)(r \to \infty) - ikW_2(r \to \infty)\right] \exp(ikr)\,.
\end{aligned} \tag{27}$$

Assume that there exists the following limit

$$U_\infty(k) = \lim_{r\to\infty} \left[-k^2 + \frac{E_2 + E_1}{2} + W_2 \frac{w_1 + w_2}{2} - ikW_2 \right] . \qquad (28)$$

Then the transformed Jost solution reads

$$f_2(k,r) = (Lf_0)(k,r) U_\infty^{-1}(k) , \qquad (29)$$

Making similar manipulations with the physical solution (23) we establish the form of transformed S-matrix

$$S_2(k) = \mathrm{e}^{\mathrm{i}l\frac{\pi}{2}} U_\infty(k) \mathrm{e}^{-\mathrm{i}l\frac{\pi}{2}} S_0(k) \mathrm{e}^{-\mathrm{i}l\frac{\pi}{2}} U_\infty^{-1}(k) \mathrm{e}^{\mathrm{i}l\frac{\pi}{2}} . \qquad (30)$$

In the case of our second-order SUSY transformation, the transformed S-matrix depends on the factorization energy $\mathcal{E}$ and parameters Q, X through the matrix multipliers $U_\infty(k)$ and $U_\infty(k)^{-1}$. In general, this dependence may be very complicated. Moreover, the scattering matrix S_2 may have unphysical low and high energy behavior.

SUSY transformations that deform the scattering matrix in a simple way are useful tools to solve inverse scattering problem. In the two-cannel case there is a special kind of deformation, when $U_\infty(k)$ becomes an orthogonal matrix [11]. We call such deformations as eigen-phase preserving transformations.

3. Eigen-phase preserving SUSY transformations

3.1. Two channel case

Let us analyze conditions that make a two-channel SUSY transformation be an eigen-phase preserving one [11]. In this case parameters of the transformation, $Q = q$, $X = x$, are just some numbers. Matrix $U_\infty(k)$ depends on q only and becomes orthogonal when $q = \pm i$. The determinant of u vanishes at large distances, $\det u(r \to \infty) \to 0$ with such choice of q. Let $\det u(r \to \infty) \simeq \epsilon$, then superpotential w diverges as $w(r \to \infty) \simeq \epsilon^{-1}$ and two-fold superpotential W_2 vanishes as $w_2(r \to \infty) \simeq \epsilon$. As a result, the limit (28) contains only even powers of k

$$U_\infty(k) = \lim_{r\to\infty} \left[-k^2 + \frac{E_2 + E_1}{2} + W_2 \frac{w_1 + w_2}{2} \right] . \qquad (31)$$

The cancelation of odd powers of k is a necessary condition to provide EPP SUSY transformations. In the next subsection we establish the most general form of matrix Q which leads to the vanishing limit

$$\lim_{r\to\infty} W_2 = 0 , \qquad (32)$$

for the case $N > 2$.

Parameter x is also should be fixed to provide $\det \mathrm{W}[u, u^*] \neq 0$ for all $r > 0$ which leads to a finite V_2.

3.2. Asymptotic SUSY transformation at large distances for arbitrary N

The transformation function u (19) has the following asymptotic behaviour at large distances

$$u(r \to \infty) \to u_\infty (I_N + \Lambda r^{-1} + o(r^{-1})) , \qquad u_\infty = Ae^{-iKr\Sigma} , \qquad (33)$$

where

$$A = \begin{pmatrix} I_M & -Q^T \\ Q & I_{N-M} \end{pmatrix}, \qquad \Sigma_{M,N-M} = \begin{pmatrix} I_M & 0 \\ 0 & -I_{N-M} \end{pmatrix}. \tag{34}$$

For each concrete EPP transformation N and $N-M$ are fixed, therefore we will use notation Σ instead of $\Sigma_{M,N-M}$.

The two-fold superpotential behaves asymptotically as

$$\lim_{r\to\infty} W_2 = W_{2,\infty} = (\mathcal{E}-\mathcal{E}^*)u_\infty^* \mathrm{W}[u_\infty, u_\infty^*]^{-1} u_\infty^T \tag{35}$$

$$= 2i\mathcal{E}_{\mathrm{Im}} A^* e^{iK^*r\Sigma} \mathrm{W}[u_\infty, u_\infty^*]^{-1} e^{-iKr\Sigma} A^T. \tag{36}$$

Using asymptotic (33) we see that this limit is a constant matrix

$$\begin{aligned} W_{2,\infty} &= 2i\mathcal{E}_{\mathrm{Im}} A^* e^{iK^*r\Sigma}\left[u_\infty^T\,(u_\infty^*)' - \left(u_\infty^T\right)' u_\infty^*\right]^{-1} e^{-iKr\Sigma} A^T \\ &= 2i\mathcal{E}_{\mathrm{Im}} A^* e^{iK^*r\Sigma}\left[e^{-iKr\Sigma}A^T A^*\left(e^{iK^*r\Sigma}\right)' - \left(e^{-iKr\Sigma}\right)' A^T A^* e^{iK^*r\Sigma}\right]^{-1} e^{-iKr\Sigma} A^T \\ &= 2\mathcal{E}_{\mathrm{Im}} A^*\left[K^* A^T A^* \Sigma + K\Sigma A^T A^*\right]^{-1} A^T\,. \end{aligned} \tag{37}$$

We introduce auxiliary matrix W_∞

$$\mathrm{W}_\infty := K^* A^T A^* \Sigma + K\Sigma A^T A^* = 2\begin{pmatrix} k_r(I_M + Q^T Q^*) & ik_i(Q^T - (Q^*)^T) \\ ik_i(Q-Q^*) & -k_r(QQ^\dagger + I_{N-M}) \end{pmatrix}. \tag{38}$$

Limit (32) leads to the following matrix equation

$$W_{2,\infty} = 0 \Rightarrow A^* \mathrm{W}_\infty^{-1} A^T = 0\,, \qquad \det \mathrm{W}_\infty \neq 0\,, \tag{39}$$

which provides asymptotic cancelation of k in (28). In the two channel case these equations fix Q uniquely. When $N>2$, these equations determine a set of Q values.

Equation (39) may be satisfied if and only if matrix A is singular. Matrix W_∞ is invertible, $\mathrm{rank}\mathrm{W}_\infty = N$. Let $\mathrm{rank}\,A = n$, then $\mathrm{rank}\,A^* = \mathrm{rank}\,A^T = n$ and $\mathrm{rank}(\mathrm{W}_\infty^{-1}A^T) = \dim\mathrm{Img}(\mathrm{W}_\infty^{-1}A^T) = n$. The dimension of kernels $\dim\mathrm{Ker}\,A = \dim\mathrm{Ker}\,A^* = \dim\mathrm{Ker}\,A^T = N-n$. Equation (39) implies that $\mathrm{Img}(\mathrm{W}_\infty^{-1}A^T) \subset \mathrm{Ker}\,A^*$, hence $n \le N-n$. Therefore equation (39) has solutions only if $n \le \frac{1}{2}N$. From the other hand, from explicit form of matrix A, (34), its rank $n \ge \max(M, N-M)$. That is, (39) has solutions if and only if

$$\mathrm{rank}\,A = \frac{N}{2}\,, \qquad N = 2M\,. \tag{40}$$

From here it follows that for odd number of channels equation (39) has no solutions.

Consider $2M\times 2M$ matrix A

$$A = \begin{pmatrix} I_M & -Q^T \\ Q & I_M \end{pmatrix}, \tag{41}$$

with $\mathrm{rank}\,A = M$. Two its rectangular sub matrices have the same rank

$$\mathrm{rank}\begin{pmatrix} I_M \\ Q \end{pmatrix} = \mathrm{rank}\begin{pmatrix} -Q^T \\ I_M \end{pmatrix} = M\,. \tag{42}$$

We can take first M columns of A as linearly independent, then from (40), (41) and (42) follows that there exists $M\times M$ matrix Z, such that

$$\begin{pmatrix} I_M \\ Q \end{pmatrix} Z = \begin{pmatrix} -Q^T \\ I_M \end{pmatrix} = M\,. \tag{43}$$

Solving this equation we obtain $Z = -Q^T$ and $QQ^T = -I_M$.

Let us extract i from Q,

$$Q = \pm iB\,, \qquad B^T B = BB^T = I_M\,, \tag{44}$$

and substitute Q in this form into (39). First of all we invert matrix W_∞. This matrix can be factorized in two ways

$$\frac{1}{2}\mathrm{W}_\infty =$$

$$\begin{pmatrix} k_r B^T(B+B^*) & -k_i(B^T+B^\dagger) \\ -k_i(B+B^*) & -k_r B(B^\dagger+B^T) \end{pmatrix} = \begin{pmatrix} k_r(B^\dagger+B^T)B^* & -k_i(B^T+B^\dagger) \\ -k_i(B+B^*) & -k_r(B+B^*)B^\dagger \end{pmatrix} =$$

$$\begin{pmatrix} k_r B^T & -k_i I_M \\ -k_i I_M & -k_r B \end{pmatrix}\begin{pmatrix} (B+B^*) & 0 \\ 0 & (B^\dagger+B^T) \end{pmatrix} =$$

$$\begin{pmatrix} (B^\dagger+B^T) & 0 \\ 0 & (B+B^*) \end{pmatrix}\begin{pmatrix} k_r B^* & -k_i I_M \\ -k_i I_M & -k_r B^\dagger \end{pmatrix}.$$

We note that

$$\mathrm{W}_\infty \mathrm{W}_\infty^* = 4(k_r^2+k_i^2)\begin{pmatrix} (B^\dagger+B^T) & 0 \\ 0 & (B+B^*) \end{pmatrix}\begin{pmatrix} (B+B^*) & 0 \\ 0 & (B^\dagger+B^T) \end{pmatrix}. \tag{45}$$

Therefore the inverse matrix reads

$$\mathrm{W}_\infty^{-1} = \frac{1}{2|K|^2}\begin{pmatrix} k_r B^\dagger & -k_i I_M \\ -k_i I_M & -k_r B^* \end{pmatrix}\begin{pmatrix} (B^T+B^\dagger)^{-1} & 0 \\ 0 & (B+B^*)^{-1} \end{pmatrix} = \tag{46}$$

$$\frac{1}{2|K|^2}\begin{pmatrix} k_r B^\dagger(B^T+B^\dagger)^{-1} & -k_i(B+B^*)^{-1} \\ -k_i(B^T+B^\dagger)^{-1} & -k_r B^*(B+B^*)^{-1} \end{pmatrix}$$

Let us introduce notations for auxiliary matrices

$$\tilde{B} = \begin{pmatrix} (B^T+B^\dagger) & 0 \\ 0 & (B+B^*) \end{pmatrix},$$

$$B_k = \begin{pmatrix} k_r B^\dagger & -k_i I_M \\ -k_i I_M & -k_r B^* \end{pmatrix}.$$

Then $W_{2,\infty} = \mathcal{E}_{\mathrm{Im}} A^* B_k \tilde{B}^{-1} A^T/(k_r^2+k_i^2)$.

Using the following matrix identities

$$B^T(B^T+B^\dagger)^{-1} = (B^*+B)^{-1}B^*\,, \qquad B^\dagger(B^T+B^\dagger)^{-1} = (B^*+B)^{-1}B\,, \tag{47}$$

$$(B^T+B^\dagger)^{-1}B^T = B^*(B^*+B)^{-1}\,, \qquad (B^T+B^\dagger)^{-1}B^\dagger = B(B^*+B)^{-1}\,, \tag{48}$$

$$B^{-1}A^T = \begin{pmatrix} (B^T+B^\dagger)^{-1} & 0 \\ 0 & (B+B^*)^{-1} \end{pmatrix}\begin{pmatrix} I_M & iB^T \\ -iB & I_M \end{pmatrix} = \tag{49}$$

$$\begin{pmatrix} I_M & iB^* \\ -iB^\dagger & I_M \end{pmatrix}\begin{pmatrix} (B^T+B^\dagger)^{-1} & 0 \\ 0 & (B+B^*)^{-1} \end{pmatrix},$$

$$A^* B_k = K^*\begin{pmatrix} B^\dagger & -iI_M \\ -iI_M & -B^* \end{pmatrix}, \tag{50}$$

we see that

$$A^* B_k \tilde{B}^{-1} A^T =$$

$$K^*\begin{pmatrix} B^\dagger & -iI_M \\ -iI_M & -B^* \end{pmatrix}\begin{pmatrix} I_M & iB^* \\ -iB^\dagger & I_M \end{pmatrix}\begin{pmatrix} (B^T+B^\dagger)^{-1} & 0 \\ 0 & (B+B^*)^{-1} \end{pmatrix} = 0\,.$$

Thus (44) gives solutions of (39).

Now we can calculate the asymptotic form of a transformation operator explicitly. To calculate asymptotic of $W_2 w$ we use the symmetry of superpotential $w = u'u^{-1} = (u^T)^{-1}(u^T)'$,

$$\lim_{r\to\infty} W_2 w = W_{2,\infty} w_\infty = 2i\mathcal{E}_{\rm Im} u_\infty^* \mathrm{W}[u_\infty, u_\infty^*]^{-1} u_\infty^T (u_\infty^T)^{-1} (u_\infty^T)' =$$

$$-2iK\mathcal{E}_{\rm Im} A^* \mathrm{W}_\infty^{-1} \Sigma A^T = \frac{-i\mathcal{E}_{\rm Im}}{K^*} A^* B_k \tilde{B}^{-1} \Sigma A^T =$$

$$2\mathcal{E}_{\rm Im} \begin{pmatrix} -iB^\dagger & I_M \\ -I_M & -iB^* \end{pmatrix} \begin{pmatrix} (B^T + B^\dagger)^{-1} & 0 \\ 0 & (B+B^*)^{-1} \end{pmatrix}$$

Matrix $W_{2,\infty}$ is real, therefore

$$\Omega = W_{2,\infty} \frac{w_\infty + w_\infty^*}{2} = \mathrm{Re}(W_{2,\infty} w_\infty) =$$

$$2k_r k_i \begin{pmatrix} i(B^T - B^\dagger) & 2I_M \\ -2I_M & i(B - B^*) \end{pmatrix} \begin{pmatrix} (B^T + B^\dagger)^{-1} & 0 \\ 0 & (B+B^*)^{-1} \end{pmatrix}$$

The matrix U_∞ defined in (28) reads

$$U_\infty(k^2) = \left(-k^2 + k_r^2 - k_i^2\right) I_N + \Omega\,, \tag{51}$$

Matrix Ω is real, orthogonal (up to a normalization), $\Omega^T\Omega = 4k_r^2 k_i^2 I_N$, and antisymmetric $\Omega = -\Omega^T$. To establish its orthogonality and antisymmetry one should use relations (47), (48). With these two properties of Ω the matrix $U_\infty(k^2)$ becomes proportional to the orthogonal matrix

$$U_\infty(k^2) U_\infty^T(k^2) = \left((-k^2 + k_r^2 - k_i^2)^2 + 4k_r^2 k_i^2\right) I_N \tag{52}$$

That is the Jost solutions at large distances are rotated by orthogonal matrix U_∞

$$(Lf)(k, r\to\infty) \to U_\infty \exp(ikr)\,, \tag{53}$$

In this case the S-matrix transformation (30) is just an energy-dependent orthogonal transformation,

$$S_2(k) = R_S(k^2) S_0(k) R_S^T(k^2)\,, \tag{54}$$

with the orthogonal matrix, $R_S^T R_S = I_N$,

$$R_S = \mathrm{e}^{\mathrm{i}l\frac{\pi}{2}} U_\infty \mathrm{e}^{-\mathrm{i}l\frac{\pi}{2}} \left[(-k^2 + k_r^2 - k_i^2)^2 + 4k_r^2 k_i^2\right]^{-1/2}\,. \tag{55}$$

That is we obtain desired generalization of two-channel EPP SUSY transformations.

The above analysis is valid for an arbitrary $M \times M$ symmetric matrix X. Transformed S-matrix S_2 depends on matrix Q only. Therefore X might provide additional $M(M+1)/2$ parametric deformation of potential V_2 without affecting the S-matrix. From the other hand, possibility of such deformations contradicts to the uniqueness of the inversion of the complete set of scattering data. Therefore, there may exist only one matrix X corresponding to one physical potential V_2. The EPP SUSY transformation should be uniquely determined by the factorization energy, $M \times M$ complex orthogonal matrix B and a sign factor. In the next subsection we show how to fix matrix X and prove that the corresponding potential V_2 is regular for all $r > 0$.

3.3. Eigen-phase preserving SUSY transformation near the origin

To analyze the properties of EPP SUSY transformation in the vicinity of $r = 0$ we will use the solution

$$\varphi_0(k,r) = \frac{i}{2k}\left[f_0(-k,r)F_0(k) - f_0(k,r)F_0(-k)\right], \tag{56}$$

vanishing at the origin

$$\varphi_0(k,r\to 0) \to \mathrm{diag}\left(\frac{r^{\nu_1+1}}{(2\nu_1+1)!!},\dots,\frac{r^{\nu_N+1}}{(2\nu_N+1)!!}\right), \tag{57}$$

where $F_0(k)$ is the Jost matrix (25). We rewrite transformation solution in the basis $(\varphi_0(K,r), f_0(K,r))$ expressing $f_0(-K,r)$ from (56)

$$f_0(-K,r) = \frac{2K}{i}\varphi_0(K,r)F_0^{-1}(K) + f_0(K,r)F_0(-K)F_0^{-1}(K), \tag{58}$$

and substituting in (19)

$$u(r) = \frac{2K}{i}\varphi_0(K,r)F_0^{-1}(K)C + f_0(K,r)(D + s_0C)\,. \tag{59}$$

where

$$s_0 = F_0(-K)F_0^{-1}(K) = \begin{pmatrix} s_1 & s_2^T \\ s_2 & s_3 \end{pmatrix}, \qquad \mathrm{e}^{\mathrm{i}l\frac{\pi}{2}} s_0 \mathrm{e}^{\mathrm{i}l\frac{\pi}{2}} = S_0\,. \tag{60}$$

We can transform matrix $(D + s_0C)$ to the form of matrix D (without affecting C) multiplying u from the right

$$(D+s_0C)\begin{pmatrix} I_M & 0 \\ -(s_2 \pm is_3B) & I_M \end{pmatrix} = \begin{pmatrix} \tilde{X} & \mp iB^T \\ 0 & I_M \end{pmatrix}, \tag{61}$$

where

$$\tilde{X} = X + s_1 \pm i(s_2^T B + B^T s_2) - B^T s_3 B, \tag{62}$$

Then the transformation solution reads

$$u(r) = \frac{2K}{i}\varphi_0(K,r)F_0^{-1}(K)\begin{pmatrix} I_M & 0 \\ \pm iB & 0 \end{pmatrix} + f_0(K,r)\begin{pmatrix} \tilde{X} & \mp iB^T \\ 0 & I_M \end{pmatrix}. \tag{63}$$

Consider the case $\tilde{X} = 0$. In this case the potential V_2 is regular for all $r > 0$. Let us prove this. According to the Wronskian representation of the second-order superpotential W_2 (16), the potential V_2 will be regular if and only if $\det \mathrm{W}[u,u^*](r) \neq 0$.

The derivative of the Wronskian $\mathrm{W}[u,u^*]$ reads

$$\mathrm{W}[u,u^*]'(r) = (\mathcal{E} - \mathcal{E}^*)u^T(r)u^*(r)\,. \tag{64}$$

By construction $\mathrm{W}[u,u^*]$ is an anti-Hermitian matrix, i.e. $\mathrm{W}[u,u^*] = -\mathrm{W}^\dagger[u,u^*]$. We represent transformation solution in a block-diagonal form

$$u(r) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \tag{65}$$

with $M \times M$ matrix blocks. When $\tilde{X} = 0$ these blocks obey the following boundary conditions: $u_{11}(0) = 0$, $u_{21}(0) = 0$, $u_{12}(\infty) = 0$, $u_{22}(\infty) = 0$. As a result the Wronskian

$$\mathrm{W}[u,u^*] = \begin{pmatrix} u_{11}^T & u_{21}^T \\ u_{12}^T & u_{22}^T \end{pmatrix}\begin{pmatrix} u_{11}'^* & u_{12}'^* \\ u_{21}'^* & u_{22}'^* \end{pmatrix} - \begin{pmatrix} u_{11}'^T & u_{21}'^T \\ u_{12}'^T & u_{22}'^T \end{pmatrix}\begin{pmatrix} u_{11}^* & u_{12}^* \\ u_{21}^* & u_{22}^* \end{pmatrix}, \tag{66}$$

has vanishing blocks at the origin, $\tilde{W}_{11}(0) = 0$, and at infinity $\tilde{W}_{22}(\infty) = 0$. Boundary behavior of $\tilde{W}_{12}$ is not determined.

Now we can calculate diagonal blocks of the Wronskian integrating its derivative

$$\frac{\mathrm{W}[u,u^*]'(r)}{(\mathcal{E}-\mathcal{E}^*)} = \begin{pmatrix} \tilde{W}'_{11} & \tilde{W}'_{12} \\ \tilde{W}'^{\dagger}_{12} & \tilde{W}'_{22} \end{pmatrix} = \begin{pmatrix} u_{11}^T u_{11}^* + u_{21}^T u_{21}^* & u_{11}^T u_{12}^* + u_{21}^T u_{22}^* \\ u_{12}^T u_{11}^* + u_{22}^T u_{21}^* & u_{12}^T u_{12}^* + u_{22}^T u_{22}^* \end{pmatrix}. \tag{67}$$

Integration of (67) with the established boundary conditions yields

$$\mathrm{W}[u,u^*](r) = (\mathcal{E}-\mathcal{E}^*) \begin{pmatrix} \int\limits_0^r \left(u_{11}^T u_{11}^* + u_{21}^T u_{21}^*\right) dt & \tilde{W}_{12} \\ \tilde{W}_{12}^{\dagger} & -\int\limits_r^{\infty} \left(u_{12}^T u_{12}^* + u_{22}^T u_{22}^*\right) dt \end{pmatrix}. \tag{68}$$

Assume that there is a point r_0 where $\det\mathrm{W}[u,u^*](r_0) = 0$. Hence, matrix $\mathrm{W}[u,u^*](r_0)$ has at least one zero eigenvalue

$$\mathrm{W}[u,u^*](r_0)\vec{v} = 0\,. \tag{69}$$

Let us represent N dimensional eigen-vector $\vec{v}$ as two M dimensional vectors $\vec{v}_u$ and $\vec{v}_d$ and rewrite (69) as a system of equations

$$\tilde{W}_{11}\vec{v}_u + \tilde{W}_{12}\vec{v}_d = 0\,, \tag{70}$$

$$\tilde{W}_{12}^{\dagger}\vec{v}_u + \tilde{W}_{22}\vec{v}_d = 0\,. \tag{71}$$

The first term of the scalar product $(\vec{v}_u, \tilde{W}_{11}\vec{v}_u) + (\vec{v}_u, \tilde{W}_{12}\vec{v}_d) = 0$, $((\vec{a},\vec{b}) = a_j^* b^j)$ is positive

$$(\vec{v}_u, \tilde{W}_{11}\vec{v}_u) =$$

$$(\vec{v}_u, \int\limits_0^{r_0} dt\left(u_{11}^T u_{11}^* + u_{21}^T u_{21}^*\right)\vec{v}_u) = \int\limits_0^{r_0} \left((u_{11}^*\vec{v}_u, u_{11}^*\vec{v}_u) + (u_{21}^*\vec{v}_u, u_{21}^*\vec{v}_u)\right) dt > 0\,,$$

therefore $(\vec{v}_u, \tilde{W}_{12}\vec{v}_d) = n_u < 0$ is real and negative. Now calculating scalar product $(\vec{v}_d, \tilde{W}_{12}^{\dagger}\vec{v}_u) + (\vec{v}_d, \tilde{W}_{22}\vec{v}_u) = 0$ with negative second term

$$(\vec{v}_d, \tilde{W}_{22}\vec{v}_d) =$$

$$-(\vec{v}_d, \int\limits_{r_0}^{\infty} dt\left(u_{12}^T u_{12}^* + u_{22}^T u_{22}^*\right)\vec{v}_d) = -\int\limits_{r_0}^{\infty} \Big((u_{12}^*\vec{d}_u, u_{12}^*\vec{v}_d) + (u_{22}^*\vec{d}_u, u_{22}^*\vec{v}_d)\Big) dt < 0\,,$$

we obtain a contradiction, $(\vec{v}_d, \tilde{W}_{12}^{\dagger}\vec{v}_u) = n_u^* = n_u > 0$. This contradiction proves that Wronskian $\mathrm{W}[u,u^*]$ have only non-zero eigenvalues for all $r > 0$. As a result $\mathrm{W}[u,u^*]$ is invertible, and hence both W_2 and V_2 are regular (finite) for all $r > 0$. Any non-zero $\tilde{X}$ will lead to the potential V_2 which is singular in some point r_0.

This prove completes our construction of multi-channel EPP SUSY transformations. In the next subsection we present an illustrative example.

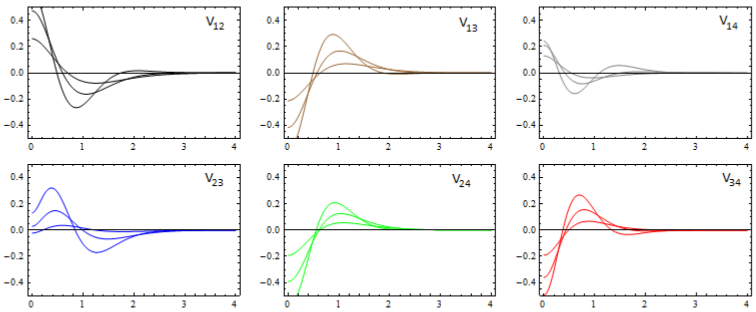

Figure 1. Off-diagonal entries of the exactly solvable potential matrix V_2 obtained from the uncoupled potential (72) with parameters $a_1 = 1.1$, $a_2 = 1.5$, $a_3 = 2.1$, $a_4 = 2.5$, $b_r = 2.5$, $b_i = 1.3$, for three choices of the factorization energy $\mathcal{E} = -2 + 1.5\mathrm{i}; -1.25 + 3.\mathrm{i}; 4.5\mathrm{i}$. The strength of coupling increases with $\arg \mathcal{E}$ decreasing from 0.78π to $\pi/2$.

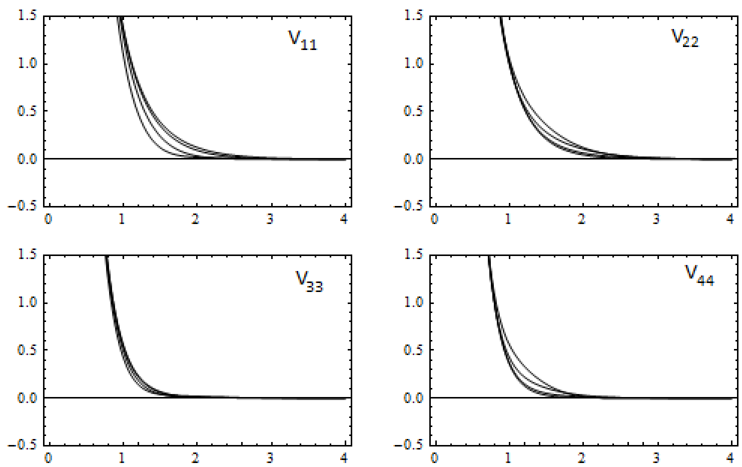

Figure 2. Diagonal entries of the exactly solvable potential matrix V_2 obtained from the uncoupled potential (72) with parameters $a_1 = 1.1$, $a_2 = 1.5$, $a_3 = 2.1$, $a_4 = 2.5$, $b_r = 2.5$, $b_i = 1.3$, for three choices of the factorization energy $\mathcal{E} = -2 + 1.5\mathrm{i}; -1.25 + 3.\mathrm{i}; 4.5\mathrm{i}$.

3.4. 4-channel coupled potential

We construct our initial 4-channel potential with $l = 0$,

$$V_0(r) = \mathrm{diag}\left[v_0(r, a_1), v_0(r, a_2), v_0(r, a_3), v_0(r, a_4)\right], \tag{72}$$

from four copies of the following single channel potential

$$v_0(r, a) = \frac{2a^2}{\sinh^2(ar)}. \tag{73}$$

Its scattering matrix is diagonal and reads

$$S_0(k) = \mathrm{diag}\left[s_0(k, a_1), s_0(k, a_2), s_0(k, a_3), s_0(k, a_4)\right], \qquad s_0(k, a) = \frac{a - ik}{a + ik}. \tag{74}$$

Consider an ingoing wave in jth channel

$$\psi_{in,j} = \exp(-\mathrm{i}kr)(\delta_{1j}, \delta_{3j}, \delta_{3j}, \delta_{4j})^T \,, \tag{75}$$

where δ_{ij} is the Kroneker delta-symbol. Ingoing wave is just a first term of long-rage asymptotic

$$\left[\mathrm{e}^{-\mathrm{i}kr} - \mathrm{e}^{\mathrm{i}kr} S_0(k)\right] (\delta_{1j}, \delta_{3j}, \delta_{3j}, \delta_{4j})^T \,. \tag{76}$$

Scattering of such wave on the potential (72) results just in a phase shift of the outgoing wave

$$\psi_{out,j} = \exp(\mathrm{i}kr + 2\mathrm{i}\delta_0(k, a_j))(\delta_{1j}, \delta_{3j}, \delta_{3j}, \delta_{4j})^T \,, \tag{77}$$

by eigen-phase

$$\delta_0(k, a_j) = -\arctan\frac{k}{a_j} \,, \tag{78}$$

without mixing between channels.

Using EPP SUSY transformation we deform potential to introduce coupling between channels. Diagonal components of the basis (φ_0, f_0) explicitly reads

$$\varphi_0(k, r; a) = \frac{1}{k^2 + a^2}\left(k\cos(kr) - a\coth(ar)\sin(kr)\right) \,, \tag{79}$$

$$f_0(k, r; a) = \exp(ikr)\frac{k + ia\coth(ar)}{k + ia} \,. \tag{80}$$

These solutions together with matrix B depending on a single complex number $b = b_r + ib_i$,

$$B = \begin{pmatrix} b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & b \end{pmatrix} , \tag{81}$$

completely define EPP SUSY transformation.

Let us fix all parameters of the model a_j, b_r, b_i except the factorization energy $\mathcal{E}$ ($a_1 = 1.1$, $a_2 = 1.5$, $a_3 = 2.1$, $a_4 = 2.5$, $b_r = 2.5$, $b_i = 1.3$). In Figures 1 and 2 we show the potential V_2 provided by EPP SUSY transformations for three values of $\mathcal{E} = -2 + 1.5\mathrm{i}; -1.25 + 3.\mathrm{i}; 4.5\mathrm{i}$. The strength of coupling increases with $\arg\mathcal{E}$ decreasing from 0.78π to $\pi/2$. One can check that $\arg\mathcal{E} = 0$ corresponds to zero-coupling, $V_2 = V_0$. For our choice of matrix B and parameters, the matrix Ω reads

$$\Omega = \mathcal{E}_{\mathrm{Im}}\begin{pmatrix} 0 & -0.936848 & 0.305791 & -0.16973 \\ 0.936848 & 0 & 0.16973 & 0.305791 \\ -0.305791 & -0.16973 & 0 & 0.936848 \\ 0.16973 & -0.305791 & -0.936848 & 0 \end{pmatrix} \tag{82}$$

The matrix S_2 (54) has the same eigenvalues, but non-diagonal character of potential results in the mixing of different channels in the outgoing wave,

$$\left[\mathrm{e}^{-\mathrm{i}kr}\mathrm{e}^{\mathrm{i}l\frac{\pi}{2}} - \mathrm{e}^{\mathrm{i}kr}\mathrm{e}^{-\mathrm{i}l\frac{\pi}{2}} R_S S_0(k) R_S^T\right] (\delta_{1j}, \delta_{3j}, \delta_{3j}, \delta_{4j})^T, \tag{83}$$

There is another set of ingoing waves

$$\psi_{in,j} = \exp(-\mathrm{i}kr)R_j(k^2) \,, \qquad R_S = (R_1, R_2, R_3, R_4) \,, \tag{84}$$

given by columns $R_j(k^2)$ of matrix R_S which scatter just with a phase shift (78),

$$\psi_{out,j} = \exp(\mathrm{i}kr + 2\mathrm{i}\delta_0(k, a_j))R_j(k^2) \,. \tag{85}$$

Vectors $\vec{R}_j$ depends on the energy of ingoing wave. In Figure 3 we show this dependence for a particular example $\mathcal{E} = 4.5\mathrm{i}$. Changing two complex parameters b and E we can manipulate transitions between channels which may open a way for broad physical application of EPP SUSY transformations.

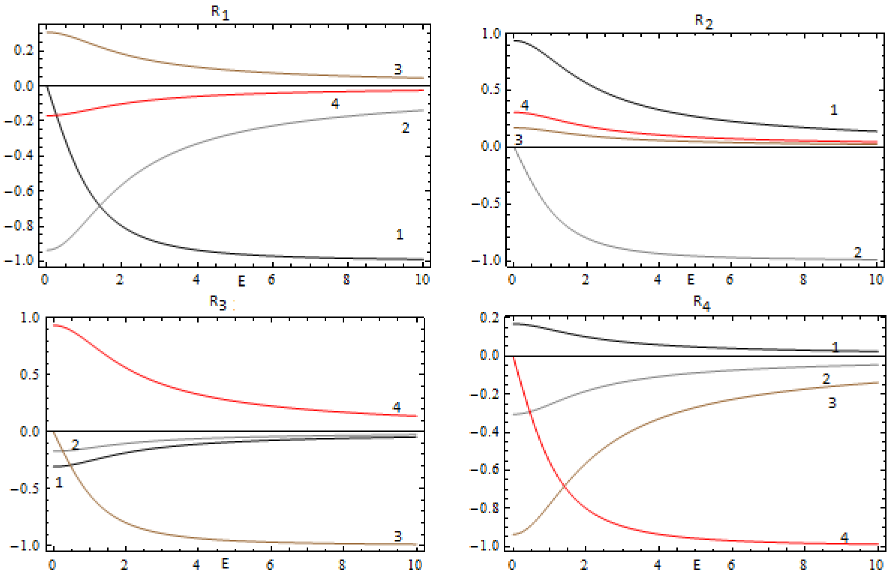

Figure 3. Eigenvectors of the scattering matrix S_2.

4. Conclusion

In the present paper we have generalized two-channel eigen-phase preserving SUSY transformations to the multichannel case, $N = 2M > 2$. It was surprising, that such generalization exists for even number of channels only. A single EPP SUSY transformation depends on a complex factorization energy $\mathcal{E}$, and $M \times M$ complex matrix B, such that $B^T B = I_N$. Therefore single EPP SUSY transformation provide an $M(M-1)+2$ parametric deformation of scattering matrix without affecting eigen-phase shifts.

There are several possible applications of presented results. One can use EPP SUSY transformations to solve inverse scattering problem by deforming a diagonal S-matrix as in [3]. We also may consider the S-matrix eigenvalues which conserved under $M(M-1)+2$ parametric deformation as integrals of motions for some dynamical system associated with matrix Schrödinger equation [4]. In this context it is interesting to establish how this dynamical system looks. We expect that in this way new exactly solvable non-linear equations may be discovered.

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